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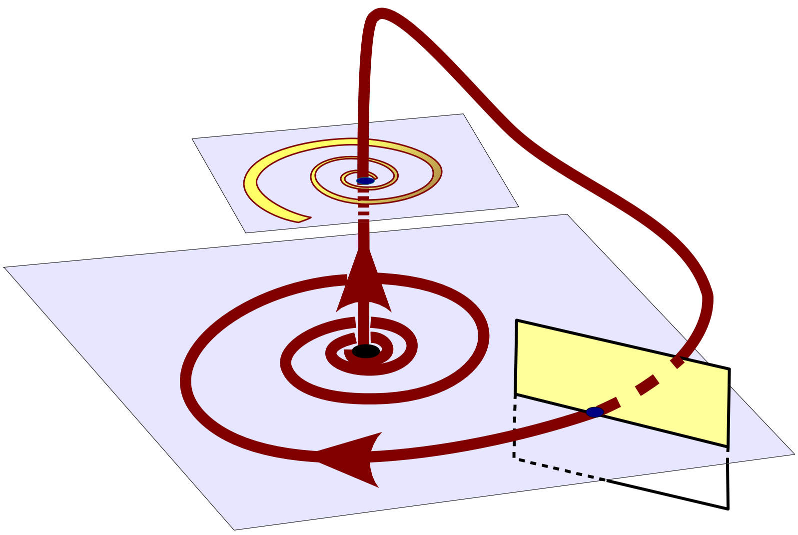
Nizhny Novgorod, 15–16 December of 2023

Book of Abstracts



Lobachevsky State University of Nizhny Novgorod

Description of the electronic issue:
Book of abstracts of the International Conference "Shilnikov Workshop 2023", held on December 15-16, 2023 at the Lobachevsky State University of Nizhny Novgorod. The book contains abstracts of plenar, sectional and poster reports devoted both to the theory of dynamical systems, the theory of bifurcations, the mathematical theory of dynamical chaos including all its three forms: conservative chaos, dissipative chaos (strange attractors) and mixed dynamics, and also to applications of the corresponding mathematical theories to various models of natural science and technology.
Electronic version of the collection of abstracts on the website:
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Conference organizer:

Scientific and Educational Mathematics Center
«Mathematics of Future Technologies»

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Contents

Smale horseshoe chains reveal the true complexity of the double scroll attractor

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The double scroll attractor is one of the most popular objects in the theory of dynamic chaos. However, most of the papers devoted to it are limited to the proof of a pair of symmetric homoclinic saddle-focus orbits and numerical experiments. The world literature still lacks rigorous proofs of the existence of a double scroll attractor, as well as a detailed analysis of its structure. In this talk we provide such proofs [1].

The central point of our talk is that the complexity of the attractor is determined by the appearance in its structure of new objects, which we called Smale horseshoe chains of period p [see Fig. 1(a)]. In particular, we prove the existence of hyperbolic Smale horseshoes of arbitrary p, including $p \to \infty$, consisting of chains of any preimages [see Fig. 1(b)]. We show that for $\nu < \frac{1}{4}$, the attractor contains chains of Smale horseshoes of infinite period, generating a hyperbolic non-wandering set conjugate to an alphabet of an infinite number of symbols.

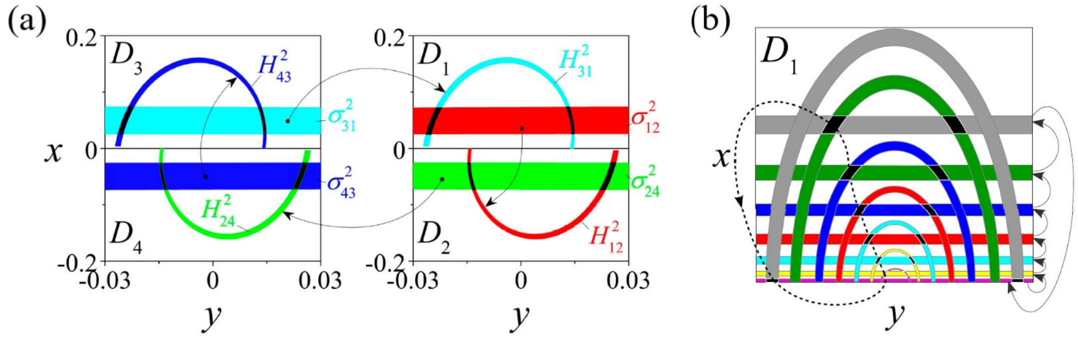

Figure 1: (a) Period-four Smale horseshoe $\sigma^2_{31} \to \sigma^2_{12} \to \sigma^2_{24} \to \sigma^2_{43} \to \sigma^2_{31}$. (b) Schematic representation of a ∞-period Smale horseshoe.

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From 1-D endomorphism to invertible multidimensional Hénon map: persistence of bifurcation structure

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The renowned 2-D invertible Hénon map turns into 1-D noninvertible quadratic map when its leading parameter b becomes zero. This well known link was studied by C. Mira who demonstrated that the bifurcation set of Hénon diffeomorphism is similar to his "box-within-a-box" bifurcation structure of 1-D endomorphism. In general, such similarity has not been strictly established, especially in multidimansional cases. In this talk we proved that the Mira bifurcation structure of a quadratic noninvertible map persists when the parameter increases from zero and the map turns into an invertible multidimensional generalized Hénon map [1]. The changes of periodic and homoclinic orbits and chaotic attractors at this transition are described. We proved the existence of Newhouse regions different from those Mira boxes that accumulate to the homoclinic bifurcations.

Acknowledgments. This work was supported by the Ministry of Science and Higher Education of the Russian Federation under Project No. 0729-2020-0036.

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On nonorientable Lorenz attractors

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We consider the one-dimensional map $\bar{x} = (-\mu + A|x|^\nu + O(|x|^{2\nu})) \cdot sign(x),\ 0 < \nu < 1$, for small values of the parameter A, also called separatrix value. From the works of L.P. Shilnikov it is known that depending on the saddle index – the parameter ν in the map under consideration – there are two fundamentally different cases: $0 < \nu \leq 1/2$ and $1/2 < \nu < 1$. In the first case, Lorenz attractors exist only for $A > 0$. In the second case, Lorenz attractors exist both for $A > 0$ and in the region $A < 0$, where the corresponding attractors are non-orientable, since when passing along the homoclinic loop of the separatrix, the two-dimensional central manifold is homeomorphic to the Möbius strip. At first, we study the boundary of the Lorenz attractor existence region for small values of the parameter A in the case $\nu > 1/2$. On the parameter plane

(B, A), where B is the coefficient in the Lorenz map before the term $|x|^{2\nu}$, the boundary of the Lorenz attractor existence region is constructed both for the cases $A > 0$, and $A < 0$. Then, the obtained results are applied for the study of the non-orientable Lorenz attractors in the Shimizu-Morioka system.

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Dynamics of a pendulum in a rarefied flow

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We consider the dynamics of a rod on the plane in a flow of non-interacting point particles moving at a fixed speed. When colliding with the rod, the particles are reflected elastically and then leave the plane of the rod and do not interact with it. The massive rod is impaled on a thin unbending weightless "knitting needle", which is attached to an anchor point and can rotate freely about it. The particles do not interact with the needle.

The equations of dynamics are obtained, which are piecewise analytic: the phase space is divided into four regions where the analytic formulas are different. There are two fixed points of the system, corresponding to the position of the rod parallel to the flow velocity, with the anchor point at front and back. It is found that the former point is topologically a stable focus, and the latter one is topologically a saddle. A qualitative description of the phase portrait of the system is obtained, namely, the phase portrait of the rod dynamics topologically coincides with the phase portrait for a mathematical pendulum with friction.

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Introduction to completely geometrically integrable maps

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The concept of the completely geometrically integrable maps is introduced. Properties of these maps are studied. Criteria of complete geometric integrability are proved. Examples are considered [1], [2].

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Chaos in Coupled Heteroclinic Cycles between Weak Chimeras

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Heteroclinic cycles are widely used in neuroscience in order to mathematically describe different mechanisms of functioning of the brain and nervous system. Heteroclinic cycles and interactions between them can be a source of different types of nontrivial dynamics. For instance, as it was shown earlier, chaotic dynamics can appear as a result of interaction via diffusive couplings between two stable heteroclinic cycles between saddle equilibria.

We go beyond these findings by considering two rotating in opposite directions coupled stable heteroclinic cycles between weak chimeras. Such an ensemble can be mathematically described by a system of six phase equations. Using two-parameter bifurcation analysis we identified regions on the plane of governing parameters where different types of nontrivial dynamics exist, including hyperchaotic, chaotic and regular spatio-temporal patterns. Note that in our numerical simulations of the system under study, we have observed that chaotic and hyperchaotic dynamics arise for small values of inter-cluster and intra-cluster coupling strengths. The increase in these coupling strength leads to a series of bifurcation transitions related to the change of symmetry of the chaotic attractors.

We assume that the origin of chaotic and hyperchaotic dynamics in similar systems is a general effect that can be observed in a range of governing parameters for heteroclinic cycles between different types of attracting sets (e.g. saddle cycles, saddle chaotic sets etc.) and various types of couplings between them (e.g. mutual synaptic couplings). Nevertheless, the verification of this hypothesis is a subject of future studies.

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The problem of particle acceleration in chromospheric solar plasma. Propagation of the Alfven wave in footpoints of magnetic loops

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The problem of particle acceleration in the solar atmosphere has been investigated for several decades. The main questions are the nature of the acceleration mechanism of these particles in solar plasma and the location of such accelerator. According to observational data [1], it can be supposed that particle acceleration can take place in dense chromospheric plasma, at footpoints of coronal magnetic loops, where various flows of magnetized plasma, arising, for example, due to Rayleigh-Taylor instability, lead to a generation of electric fields. These fields inject a large amount of cold plasma into upper hotter coronal parts of loops. This scenario may be the key to understanding the problem of solar flare initiation and explaining the abnormally large number of particles accelerated during the pulse phase.

In this paper, a model of a chromospheric accelerator, based on the action of a nonlinear Alfven pulse, which was first considered in the articles [2,3], is developed. Within the framework of the magnetohydrodynamic approximation, the propagation of the Alfven wave in an exponentially expanding coronal arch is studied: a generalized wave equation (1) is obtained and its solution is found, and an estimate of the emerging nonlinear electric field leading to particle acceleration is also given.

$$\frac{\partial^2 U}{\partial \tau^2} = x^2 \frac{\partial^2 U}{\partial x^2} + 2x \frac{\partial^2 U}{\partial x \partial y} + \frac{\partial^2 U}{\partial y^2} + 4x \frac{\partial U}{\partial x} + 3 \frac{\partial U}{\partial y} + 2U \tag{1}$$

Here U is a dimensionless component of the Alfvenic wave's magnetic field, τ is dimensionless time, x and y are radial and vertical dimensionless coordinates of the arch.

Acknowledgments. This work was supported by the Russian Science Foundation, Grant No.22-12-00308.

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PDE **entire orbits are not** ***complex*** **entire: a nonlinear example**

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In parabolic or hyperbolic PDEs, solutions which remain uniformly bounded for all real times $t = r \in \mathbb{R}$ are often called *PDE entire*. For example, consider the parabolic PDE

$$w_t = w_{xx} - 6w^2 + \lambda\,,$$

$0 < x < \frac{1}{2}$, under Neumann boundary conditions. By its gradient-like structure, all *real entire* non-equilibrium orbits $\Gamma(r)$ are heteroclinic among equilibria $w = W_n(x)$.

We show that the complex time extensions $\Gamma(r + \mathrm{i}s)$ of the real analytic heteroclinics, in contrast, are *not complex entire*. For suitable r_0, in particular, the reversible complex-valued solutions $\psi(s) := \Gamma(r_0 - \mathrm{i}s)$ of the nonlinear and nonconservative Schrödinger equation

$$\mathrm{i}\psi_s = \psi_{xx} - 6\psi^2 + \lambda$$

blow up in the imaginary time direction, i.e. for some finite real $\pm s$.

Our result is based on Poincaré non-resonance of unstable eigenvalues at W_n, near pitchfork bifurcation. For technical reasons, we have to except a discrete set of $\lambda > 0$, and are currently limited to unstable dimensions $n \leq 22$.

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On bifurcations of quadratic homoclinic tangencies in families containing a nonhyperbolic fixed point

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Consider a C^r-diffeomorphism f_0 satisfying conditions (A)–(D) below.

A) f_0 has a fixed point O with multipliers $\nu_1 = 1, \nu_2 = \lambda$, where $|\lambda| < 1$.

It is well known that the restriction of f_0 onto the local (one-dimensional) center manifold of O can be written either in the form

$$\bar{y} = y + L_n y^{n+1} + o(y^{n+1}),$$

where $L_n \neq 0$, or in the form

$$\bar{y} = y + o(y^r).$$

Then, as is usually defined, the non-hyperbolic point O has the degeneracy of order n in the first case (and $L_n \neq 0$ is called the Lyapunov value), and it has the degeneracy of indefinite or infinite order in the second case.

B) The fixed point O has the degeneracy of order $n \geq 1$ and $L_n > 0$.

Conditions A)–B) mean that the point O has a saddle-node type for odd n or a weak saddle type for even n (it is a nondegenerate saddle-node for $n = 1$ and a nondegenerate weak saddle for $n = 2$). It is well known that two local C^r-smooth one-dimensional invariant manifolds, unstable and strong stable, pass through the point O. We assume that these manifolds can be extended to the global ones and

C) For diffeomorphism f_0, the global unstable W^u and strongly stable W^{ss} manifolds of the point O are tangent quadratically at the points of a homoclinic orbit Γ_0.

We embed the diffeomorphism f_0 in a one-parameter family f_μ such that

D) at varying the parameter μ, the initial homoclinic tangency splits generally but the degeneracy type of the fixed point O as preserved.

We will study bifurcations in the family f_μ, thus, we deal with the problem on (homoclinic) bifurcations in the class of two-dimensional diffeomorphisms having a non-hyperbolic fixed point.

The following theorem describes the main bifurcations of single-round periodic orbits in the family f_μ , i.e., such orbits that have periods k, where $k = \bar{k}, \bar{k}+1, ...$, and go around the neighborhood $U(O \cup \Gamma_0)$ exactly once by the period.

Theorem. Let f_μ be a one-parameter family under consideration: the diffeomorphism f_0 satisfies conditions A),B) and C) and the family f_μ splits the initial tangency due to condition D). Then, in the segment $(-\mu_0, \mu_0)$ for any $\mu_0 > 0$ there exists a countable set of disjoint intervals δ_k, $k = \bar{k}, \bar{k}+1, ...$, of values of μ that accumulate to $\mu = 0$ as $k \to \infty$ and such that the diffeomorphism f_μ at $\mu \in \delta_k$ has an asymptotically stable single-round orbit of period k. The boundaries of intervals δ_k are points μ_k^+ and μ_k^- corresponding to nondegenerate saddle-node and period doubling bifurcations, respectively.

Remark. We show that the intervals δ_k have lengths of the order $k^{-2-2/n}$, decreasing according to a power law as $k \to \infty$. This gives them the following orders: k^{-4} for $n = 1$ (nondegenerate saddle node); k^{-3} for $n = 2$ (nondegenerate weak saddle); $k^{-8/3}$ for $n = 3$, etc, with the limit k^{-2} as $n \to \infty$. Note that in the case of a quadratic homoclinic tangency to a hyperbolic saddle, [1], the lengths of the corresponding intervals δ_k change exponentially: $l(\delta_k) \sim \gamma^{-2k}$, where $\gamma, |\gamma| > 1$, is the unstable multiplier of the saddle.

Acknowledgments. The work is supported by the Basic Research Program at HSE University.

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On orientable and non-orientable strange attractors in three-dimensional maps with axial symmetry

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We select a class of three-dimensional maps with the axial symmetry $\{x \to -x, y \to -y, z \to z\}$ and the constant Jacobian. We study bifurcations and chaotic dynamics in quadratic 3D maps from this class and show that these maps can possess discrete Lorenz-like attractors of various types. We give a description of bifurcation scenarios leading to such attractors and show examples of their implementation in our maps. We also describe the case when map has negative Jacobian and study non-orientable strange attractors.

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On Shilnikov chaos in the model of a tumor growth

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We study a model of cancer tumor growth based on the biological interactions between the cells of three types: normal, immune and tumor ones. This is so-called de Pillis-Radunskaya model proposed in [1,2] takes into account the idea of a chaotic type of the dynamics for tumor growth in a host environment, clinically observed temporal oscillation in tumor size and tumor dormancy.

$$\begin{cases} \dot{x}_1 = x_1(1-x_1) - a_{12}x_1x_2 - a_{13}x_1x_3, \\ \dot{x}_2 = r_2x_2(1-x_2) - a_{21}x_1x_2, \\ \dot{x}_3 = \frac{r_3x_1x_3}{x_1+k_3} - a_{31}x_1x_3 - d_3x_3 \end{cases}$$

The phase space in this system is specified by the variables x_1 – a population of cancer cells, x_2 – a population of normal cells and x_3 – a population of immune cells. The detailed description of the biological meaning of the parameter set can be found in [1, 2].

The existence of Shilnikov chaos associated with the emergence of a loop of a saddle-focus equilibrium state was established in [2]. We extend these results and find out the consequences of bifurcations from the periodic dynamics to Shilnikov attractors in the framework of well-known Shilnikov scenario [3]. It is important to note that the this scenario relates to two codimension-two bifurcations with either a single (Zero-pair bifurcation) or a double zero eigenvalues (Bogdanov-Takens bifurcation) with the presence of the addition degeneracy when two equilibria of the model under consideration lie on an invariant coordinate axis. We study the unfoldings of such bifurcations and find out how they are connected with Shilnikov attractors and torus-chaos.

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On genuinely discrete Lorenz attractors in 3D maps

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The Lorenz attractor is a strange attractor first found by E. Lorenz in a system of three differential equations [1]. The theory of the Lorenz attractor was laid down in the works of Afraimovich-Bykov-Shilnikov [2], [3]. According to this theory, Lorenz attractors can be pseudohyperbolic (each trajectory on the attractor is unstable and this property persists under small perturbations of the system). For the Lorenz system this was proved in [4]. The Lorenz attractor has a discrete analog, which was first found and investigated in [5]. In [5] it was also suggested that such in [6] attractor is pseudohyperbolic, but numerical studies were carried out later. Note that this attractor has zero second Lyapunov exponent, which makes it similar to the flow case in some sense. In this talk we present new example of discrete attractors of Lorenz type.

In this paper we obtain a discrete map by integrated the Lorenz system by the implicit Euler method:

$$\begin{cases} \bar{x} = x + \delta(\sigma(y - x)), \\ \bar{y} = y + \delta(\bar{x}(\rho - z) - y), \\ \bar{z} = z + \delta(\bar{x}y - \beta z). \end{cases} \tag{1}$$

with parameters $\beta = 8/3, \sigma$, ρ and integration step δ. When $\delta \to 0$ the dynamics of the map resembles that of the Lorenz system. Moreover, for δ small enough the map (1) becomes a near-the-identity map, then it can be formally embedded into a 3D time-periodic flow (a suspension) defined in a suitable topological manifold.

With an increase in the integration step, a new discrete Lorenz attractor was found, the second Lyapunov exponent of which is clearly distinguished from zero. Thus, such an attractor has no flow analogue. In addition, we show that this attractor is pseudohyperbolic.

In the same map we found non-orientable discrete Lorenz attractor, similar to the flow attractor in the Shimizu-Morioka system. The new attractor is also pseudohyperbolic.

Acknowledgments. The work is supported by the Basic Research Program at HSE University.

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Route to chaos via a cascade of period-doubling bifurcations without parameters

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Usually, bifurcations exhibited by nonlinear dissipative systems occur when varying the systems parameters. In contrast, the period-doubling bifurcations discussed in the current research are induced by changing the initial conditions whereas parameter values are fixed. Thus, the studied bifurcations can be classified as the period-doubling bifurcations without parameters. Bifurcations without parameters are characterized by a continuous dependence of the system dynamics on initial conditions at fixed parameter values. Such bifurcations are typical for oscillators with manifolds of non-isolated limit sets such as lines or surfaces of equilibria, attractive manifolds of non-isolated closed curves, etc. In the present report, we demonstrate the period-doubling bifurcations without parameters by means of numerical modelling on an example of a modified Anishchenko-Astakhov self-oscillator with a line of equilibria where the ability to exhibit bifurcations without parameters is associated with the properties of a memristor included into the model circuit. Moreover, we demonstrate that a cascade of such bifurcations caused by continuous varying initial conditions results in the chaotic dynamics and represents a particular kind of the Feigenbaum scenario.

In addition, we analyse how the memristor properties can affect the observed phenomena. In particular, it has been established that oscillatory regimes in the model with the ideal memristor are extremely sensitive to inaccuracies, internal dynamic noise and external perturbations. Nevertheless, one can detect the manifestations of the period-doubling bifurcations without parameters in transients of the system involving the memristor with forgetting effect. The discussed results are presented in paper [1].

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Twin heteroclinic connections of reversible systems

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We examine a smooth four-dimensional vector fields reversible under some smooth involution L that has a smooth two-dimensional submanifold manifold of fixed points. Our main interest here is about the orbit structure of such system near two types of heteroclinic connections involving saddle-foci and heteroclinic orbits connecting them. The first of twin heteroclinic connections in a reversible system is a connection which contains two asymmetric saddle-foci and two symmetric non-degenerate heteroclinic orbits connecting these two saddle-foci. Another type of twin heteroclinic connections in a reversible system is a connection which contains two symmetric saddle-foci and two asymmetric heteroclinic orbits which join saddle-foci and are permuted by the involution. In both cases we found families of symmetric periodic orbits, multi-round heteroclinic connections and countable families of homoclinic orbits of saddle-foci. All this say about very complicated nearby orbit structure. An example of non-variational stationary Swift-Hohenberg equation is considered where such structure has been found numerically.

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Local bifurcations in the convective Cahn-Hilliard-Oono equation

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Consider the following nonlinear evolution equation

$$u_t + u_{xxxx} + bu_{xx} + au + b_3(u^3)_{xx} + a_2(u^2)_x = 0, \tag{1}$$

where $u = u(t, x), a, b, a_2, b_3 \in \mathbb{R}$ и $a \geq 0$. If $a_2 = a = 0$, then we get the classic version of the equation known as "Cahn-Hilliard equation" [1]. If $a_2 = 0, a > 0$, then this option is usually called the Cahn-Hilliard-Oono equation. Finally, when $a_2 \neq 0$, convection is already taken into account in this equation [2,3].

This equation should be supplemented with boundary conditions. The most common version of such conditions is periodic boundary conditions

$$u(t, x + 2\pi) = u(t, x), x \in \mathbb{R} \tag{2}$$

or homogeneous Neumann boundary conditions

$$u_x(t, 0) = u_x(t, \pi) = u_{xxx}(t, 0) = u_{xxx}(t, \pi) = 0. \tag{3}$$

Boundary value problems (BVP) (1), (2); (1), (3) are given in renormalized form.

The report at the "Shilnikov Workshop" in 2022 was devoted to the boundary value problem (1), (2). This report will discuss about BVP (1),(3) and let $a > 0, c > 0$. The case $a = c = 0$ leads to the classical version of the Cahn-Hilliard equation and will be considered separately.

Let $a = a_m(\delta) = m^2(m + \delta)^2, b = b_m(\delta) = m^2 + (m + \delta)^2, \delta \in (-1.0) \cup (0, 1)$, then in the problem of the stability of the zero solution of BVP (1),(3) the critical case of a simple zero eigenvalue is realized. At

$$a = a(\delta)(1 - \alpha_1\varepsilon),\ b = b(\delta)(1 + \alpha_2\varepsilon),\ \varepsilon \in (0, \varepsilon_0),\ \alpha_1, \alpha_2 \in \mathbb{R}$$

the close to critical case of a simple zero eigenvalue is realized. In this case, the nature of the bifurcations depends on the parity of m.

If $m = 2k - 1$, then the BVP (1),(3) is characterized by transcritical bifurcations. If $a = 2k$ bifurcations of non-zero equilibrium states of the "fork" type are realized. In both cases, asymptotic formulas are derived, but with different asymptotics.

Let $a = m^2(m + 1)^2, b = m^2 + (m + 1)^2$. Then the stability spectrum of the BVP (1),(3) contains a double zero eigenvalue. When realizing the case of a close to critical double zero eigenvalue, an analysis of the neighborhood of the zero equilibrium state is given and the corresponding bifurcations are identified.

When $a = c = 0$, a special version of the bifurcation problem arises when, in the neighborhood of a homogeneous equilibrium state, spatially inhomogeneous equilibrium states arise, forming invariant manifolds of dimension 2.

The results related to the study of the boundary value problem (1), (3) are presented in work [4], and the results of the analysis of the boundary value problem (1), (2) are published in work [5].

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Quasilinear approach to the nonlinear stage of the Weibel instability in collisionless plasma

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For various transient processes in the nonequilibrium cosmic and laboratory plasmas, the nonlinear evolution of a spatial spectrum of the Weibel instability, which is caused by an anisotropy of the particle velocity distribution and results in the formation of quasi-magnetostatic turbulence, is important. A spectral quasilinear approach to the evolution of this turbulence in collisionless plasma has been developed, which takes into account only the integral nonlinear interaction of modes through their joint change of the homogenuous component of the particle velocity distribution [1]. In this approximation a closed system of equations is obtained for the two-dimensional evolution of spatial modes (harmonics) of the particle distribution function and the electromagnetic field under conditions when the plasma anisotropy axis, the wave vector and the magnetic field of the modes are mutually orthogonal each other:

$$\frac{\partial \psi_0}{\partial \tau} + \sum_{n=1}^{m} \mathrm{Re}\left[\hat{\Theta}_1\left(e_{\vec{K}_{\vec{n}}}, \vec{b}_{\vec{K}_{\vec{n}}}, \psi^*_{K_n}\right)\right] = 0, \tag{1}$$

$$\frac{\partial \psi_{K_n}}{\partial \tau} + \mathrm{i}\left(K_{\vec{n}}\right)_x \beta_x \psi_{K_n} + \mathrm{i}\left(K_{\vec{n}}\right)_z \beta_z \psi_{K_n} + 2\hat{\Theta}_1\left(e_{\vec{K}_{\vec{n}}}, \vec{b}_{\vec{K}_{\vec{n}}}, \psi_0\right) + \hat{\Theta}_1\left(e^*_{\vec{K}_{\vec{n}}}, \vec{b}^*_{\vec{K}_{\vec{n}}}, \psi_{2K_n}\right) = 0, \tag{2}$$

$$\frac{\partial \psi_{2K_n}}{\partial \tau} + 2\mathrm{i}\left(K_{\vec{n}}\right)_x \beta_x \psi_{2K_n} + 2\mathrm{i}\left(K_{\vec{n}}\right)_z \beta_z \psi_{2K_n} + \hat{\Theta}_1\left(e_{\vec{K}_{\vec{n}}}, \vec{b}_{\vec{K}_{\vec{n}}}, \psi_{K_n}\right) = 0, \tag{3}$$

$$\frac{\partial \left(b_{\vec{K}_{\vec{n}}}\right)_z}{\partial \tau} = -\mathrm{i}\left(K_{\vec{n}}\right)_x e_{\vec{K}_{\vec{n}}}, \quad \frac{\partial \left(b_{\vec{K}_{\vec{n}}}\right)_x}{\partial \tau} = \mathrm{i}\left(K_{\vec{n}}\right)_z e_{\vec{K}_{\vec{n}}}, \tag{4}$$

$$\frac{\partial e_{\vec{K}_{\vec{n}}}}{\partial \tau} = \mathrm{i}\left(b_{\vec{K}_{\vec{n}}}\right)_x \left(K_{\vec{n}}\right)_z - \mathrm{i}\left(b_{\vec{K}_{\vec{n}}}\right)_z \left(K_{\vec{n}}\right)_x + \beta_{\parallel 0}^{-1} \iiint\limits_{-\infty}^{+\infty} \beta_y \psi_{\vec{K}_{\vec{n}}}(\tau, \beta_x, \beta_y, \beta_z) d\beta_x d\beta_y d\beta_z, \tag{5}$$

$$\hat{\Theta}_1\left(e_{\vec{K}_{\vec{n}}}, \vec{b}_{\vec{K}_{\vec{n}}}, \psi(\beta)\right) = \beta_{\parallel 0} \frac{e_{\vec{K}_{\vec{n}}}}{2} \frac{\partial \psi(\beta)}{\partial \beta_y} - \beta_{\parallel 0} \frac{\left(b_{\vec{K}_{\vec{n}}}\right)_z}{2} \left(\beta_x \frac{\partial \psi(\beta)}{\partial \beta_y} - \beta_y \frac{\partial \psi(\beta)}{\partial \beta_x}\right) - \beta_{\parallel 0} \frac{\left(b_{\vec{K}_{\vec{n}}}\right)_x}{2} \left(\beta_z \frac{\partial \psi(\beta)}{\partial \beta_x} - \beta_x \frac{\partial \psi(\beta)}{\partial \beta_z}\right). \tag{6}$$

Definition of variables and parameters involved in the system of equitions (1)-(6) is given in [1]. In a wide range of both small and large values of initial electron anisotropy, the equations are solved numerically for many hundreds of modes with the help of Leapfrog method.

As a result, nonlinear evolution of modes is established, the typical patterns of deformation (flattening) of the particle velocity distribution function are picked out and characteristic evolution of both the electron anisotropy parameter and magnetic field energy are investigated. Based on the comparison of the obtained results with the solution of the same problem by means of the correct particle-in-cell code EPOCH, the contribution of various nonlinear effects to the evolution of the spatial spectrum of Weibel turbulence is identified and its properties, including the self-similar nature and four qualitatively different stages of the dynamics of unstable modes, is studied.

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Симплектические частично-гиперболические автоморфизмы на $\mathbf{T^6}$

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Изучаются автоморфизмы 6-мерного тора $\mathbf{T^6}$, порожденные целочисленными унимодулярными матрицами. Тор $\mathbf{T^6}$ снабжается симплектической структурой, заданной кососимметрической невырожденной целочисленной матрицей в $\mathbf{R^6}$ и предполагается, что матрица A, задающая автоморфизм f_A, является симплектической относительно симплектической структуры, частично-гиперболической с простыми собственными значениями. Частичная гиперболичность означает, что собственные значения матрицы A лежат как вне единичной окружности, так и на ней. Поэтому могут быть два основных случая: 1) вне единичной окружности лежат два собственных значения – вне и внутри окружности, и четыре простых собственных значения лежат на единичной окружности и 2) вне единичной окружности лежат четыре собственных значения, действительные или комплексно сопряженные, и пара комплексно сопряженных собственных значений лежат на единичной окружности. В первом случае на 6-мерном торе порождается слоение на одномерные неустойчивые слои (а также слоение на одномерные устойчивые слои), а во втором случае соответствующие слои двумерны. Основной вопрос исследования – топологическая структура слоения на неустойчивые слои автоморфизма и классификация таких автоморфизмов относительно соотношения топологической эквивалентности. Автоморфизм может порождать либо транзитивное слоение на неустойчивые слои, либо замыкание каждого слоя является тором меньшей размерности (разложимый случай). Классификация дается для всех случаев.

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Coexistence of Strange Atractors in Unidirectionally Coupled Maps

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One way to classify dynamic systems is to divide them into systems with regular and strange dynamics [1]. Usually the strange dynamics is associated with a chaotic attractor with a non-integer fractal dimension [2] and positive Lyapunov exponent. [3]. However there are "strange" attractors with a non-integer fractal dimension which are not chaotic since all its Lyaponov exponents are non-positive [4]. Most of these attractors arise in the systems with quasiperiodic driving, although their appearance in the systems with periodic or random forcing is not denied [5].
In this work we consider a system that consists of particle moving between two walls, one of which is plane and fixed and the other one is harmonically corrugated and oscillates harmonically [6]. By some simplifications we obtain a system of two unidirectionally coupled 2D maps [7]:

$$\begin{cases} \alpha_{n+1} = \alpha_n - 2C \sin\phi_n \\ \phi_{n+1} = \phi_n + A \tan\alpha_{n+1} \end{cases} \qquad (1)$$

$$\begin{cases} \Omega_{n+1} = \Omega_n + 2B \sin\psi_n \cos\alpha_{n+1} \\ \psi_{n+1} = \psi_n + \frac{1}{\Omega_n + 2B \sin\psi_n} \end{cases} \qquad (2)$$

The master system (1) is Tennyson-Lieberman-Lichtenberg system [8] with fixed walls and the slave system (2) is Ulam-like map driven by the master system. The Tennyson-Libermann-Lichtenberg system [8] is conservative and has regions of invariant (KAM) tori; for this reason, the master system can provide quasiperiodic driving.
We show that with the irrational ratio of the parameters B and C of the system, various

regimes can arise, including strange non-chaotic attractors. Spectral analysis and analysis of Lyapunov exponents were carried out to confirm the type of regime. The important result is that a large number of different strange non-chaotic attractors coexists in the system.

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Estimates of the chain recurrent sets for generalized Hénon maps

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A class of polynomial maps on $\mathbb{R}^m$ or $\mathbb{C}^m$ which is defined by the assumption that the difference equation induced by the map under consideration generates a polynomial with leading monomial of a single variable. We show that the chain recurrent set of such a map is bounded and provide estimates on its projections to the coordinate axis in terms of maximal positive roors of the induced polynomial. As for unbounded orbits, it turns out that some kind of monotonicity for the wandering orbits takes for any map from this class when passing to infinity.

The class under consideration is proved to contain the generalized Hénon maps and the Arneodo-Coullet-Tresser maps. In particular, the generalized Hénon maps of the form

$$f = f_{g,b} : \mathbb{R}^2 \to \mathbb{R}^2 \ \text{ with } \ f(x, y) = (y + g(x), -bx),$$

where $b \neq 0$ and g is a polynomial of degree $k > 1$, belong to this class. By using the normal forms one may assume that $g(x) = \pm x^k + O(x^{k-2})$. As a corollary, the chain recurrent set of such generalized Hénon map is contained in the box $[-M_1, M_1] \times [-M_2, M_2]$, where M_1, M_2 are the maximal positive roots of induced polynomials with respect to the first and second variables, respectively.

We also discuss relations of this result to multidimensional perturbations [1].

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Polariton modes of a superradiant laser with combined Fabry—Perot cavity equipped with distributed feedback

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We consider polariton modes in superradiant lasers [1,2,3] where the photon lifetime, T_E, in the cavity is less than the relaxation time of the polarization of an active medium (optical dipole oscillations of the active centers), T_2. In such lasers, low-Q cavities make impossible lasing of electromagnetic modes. Previously, the properties of polariton modes were studied mainly for media without population inversion of the energy levels of active centers, while in the case of superradiant lasers it is necessary to study the dispersion properties and lasing thresholds of the polariton modes in active media with population inversion.

In this work, a detailed analytical and numerical study of the properties of polariton modes is carried out for a superradiant laser with a low-Q combined cavity, namely, a Fabry—Perot cavity with distributed feedback (DFB) of counterpropagating waves. For this purpose, we use the characteristic and dispersion equations obtained from the Maxwell—Bloch system of equations [1] within an approximation of a homogeneous spectral and spatial population inversion of the levels of active centers (in the absence of a nonlinear population inversion grating) under appropriate boundary conditions at the ends of the cavity. In particular, the behavior of eigenfrequencies and growth/decay rates of polariton modes is studied as functions of the population inversion and DFB coefficient (Fig. 1). Change of the latter makes it possible to control the width of the photonic band gap, where electromagnetic waves do not propagate and are attenuated, but unstable polariton modes exist. The lasing thresholds for various modes are determined.

Thus, it is shown that, in a certain range of laser parameters, the unstable polariton modes can exist inside the photonic band gap. Also, on the basis of numerical modelling of the Maxwell—Bloch equations, typical regimes of lasing are analyzed.

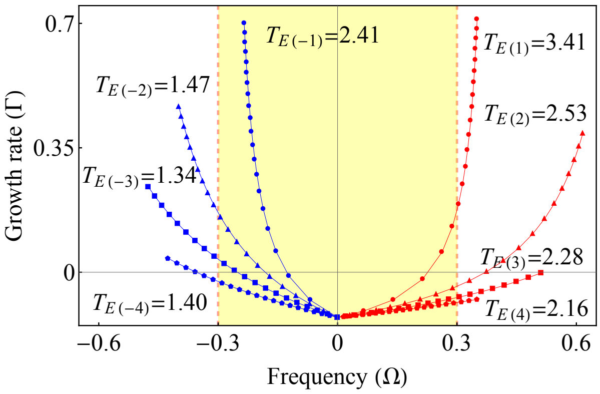

Figure 2: Dependences of the frequency detunings and growth rates of polariton modes on the population inversion of active centers in a combined cavity with length $L = 4$, mirror reflection factors $R = 0.1$, DFB coefficient $\beta = 0.3$, polarization relaxation time $T_2 = 8$. The values are normalized according to [1,2]. A vertical yellow stripe indicates the photonic band gap. The sequences of points are given for the values of population inversion ranging from 0 to 1 with a step of 0.05

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Parametric perturbations of two–dimensional Hamiltonian systems with nonmonotonic rotation

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We study nonconservative quasi–periodic (with m frequencies) perturbations of two–dimensional Hamiltonian systems with nonmonotonic rotation. It is assumed that the perturbation contains the so–called *parametric* terms. The behavior of solutions in the vicinity of degenerate resonances is described. Conditions for the existence of resonance $(m+1)$–dimensional invariant tori, for which there are no generating ones in the unperturbed system are found. The class of perturbations for which such tori can exist is indicated. The results are applied to the asymmetric Duffing equation under a parametric quasi–periodic perturbation.

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Dynamical regimes in networks of FitzHugh-Nagumo oscillators under noise influence

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Due to the fact that real systems are always networks of interacting subsystems, studying dynamics of coupled elements is a topical problem in nonlinear science and its applications. Previously, it was believed that only different types of synchronization and desynchronization could be established in these ensembles. However, at the beginning of the twentieth century, special types of partial synchronization were discovered. These are chimera states and solitary states. A chimera state is defined as a spatiotemporal pattern in which an array of identical oscillators are split into coexisting regions of coherent (phase and frequency locked) and incoherent (drifting) oscillations. Solitary states are regimes which are associated with network states for which single or several elements behave differently compared with neighbouring elements which can behave coherently or can be completely synchronized. These partial synchronization patterns were also observed in a wide range of experimental systems but also in real-world systems. Chimera states can be linked to various processes occurring in real-world systems, for example, in power-grids networks, neuroscience, in social systems, and so on. Solitary states can also be found in neural networks when we try to analyze dynamics of a single cell. With the discovery of chimeras and solitary states, the interest in the study of the noise effect on the behavior of complex nonlinear systems has significantly increased.

Our work deals with the influence of additive noise with normal (Gaussian noise) and anomalous (Lévy noise) on the dynamics of a ring of nonlocally coupled FitzHugh-Nagumo oscillators [1,2]. It was previously shown that depending on coupling parameters between the individual elements this network can demonstrate various spatiotemporal structures, such as chimera states, solitary states and regimes of their coexistence (combined structures), synchronization, incoherence, and traveling waves [3]. The studies carried out in this work have shown that these patterns exhibit different responses against additive noise influences. It is shown that such sources of additive noise lead to the suppression of solitary states and combined states in the FitzHugh-Nagumo network, while Lévy noise requires a lower noise intensity to suppress these structures. When adding noise sources into the neural network at the control parameters corresponding to the regime of multistability, the probability of establishing chimera states increases up to 100%, although without noise their probability tends to 0%.

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Strange Attractors in the Model of Neuron-Astrocyte Interaction

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In this study we propose a new reduced phenomenological model that allows us to reproduce the bursting mean-field activity of neuronal populations. We supplemented the Tsodyks-Markram model with the additional equation in order to describe main features of the neuron-glial interactions. This model is a simplification of the previously proposed full model of tripartite synapse, which can be mathematically described as follows:

$$\begin{cases} \tau\dot{E} = -E + \alpha\ln\left(1 + \exp\frac{JU(y)xE+I_0}{\alpha}\right) \\ \dot{x} = \frac{1-x}{\tau_D} - U(y)xE \\ \dot{y} = \frac{-y}{\tau_Y} + \beta\sigma_y(x) \end{cases} \quad (1)$$

Here $U(y) = U_0 + \frac{\Delta U_0}{1+e^{-50(y-y_{thr})}}$, and $\sigma_y(x) = \frac{1}{1+e^{-20(x-x_{thr})}}$. The variables have the following meanings: $E(t)$ – the average population activity, $x(t)$ – the amount of available neurotransmitter, and $y(t)$– the concentration of gliotransmitter released as a result neuron-astrocyte interaction.

In this study we choose I_0 and U_0 as control parameters of the model. All other parameters are set as follows: $\Delta U_0 = 0.305$, $\tau_D = 0.07993$, $\tau_y = 3.3$, $\tau = 0.013$, $\beta = 0.3$, $x_{thr} = 0.75$, $y_{thr} = 0.4$, $J = 3.07$, $\alpha = 1.58$.

It was shown that spiral attractors can arise in this system according to the Shilnikov scenario, the last step of which leads to the appearance of a homoclinic attractor containing a homoclinic loop to the saddle-focus equilibrium with a two-dimensional unstable manifold. As a result, the system under study can generate several types of bursting population activity with different properties.

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Double-sided bound of the solution of the Dirichlet problem for the Laplace equation on a circle

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Let us consider the Dirichlet problem for the Laplace equation on a unit circle:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0\,, \qquad u|_{r=1} = f(\theta)\,, \quad (1)$$

where r and θ are polar coordinates on the plane.

It is well known that exact solution of problem (1) is expressed by means the Poisson formula [1]:

$$u(r,\theta) = \frac{1}{2\,\pi} \int_0^{2\,\pi} \frac{1-r^2}{1-2\,r\,\cos(\theta-\varphi)+r^2}\, f(\varphi)\, d\varphi\,. \tag{2}$$

However, in practice, the integral (2) for general functions $f(\theta)$ cannot be calculated explicitly. Nevertheless, in some cases it is possible to overcome this obstacle and obtain some information about the spatial structure of the function (2) using the next statement.

Theorem. *Let $f(\theta) > 0$ under $\theta \in [0, 2\,\pi]$, then exact solution $u(r,\theta)$ of the problem (1) on the open unit circle obeys to the following double-sided bound:*

$$u^-(r) \le u(r,\theta) \le u^+(r)\,, \tag{3}$$

where

$$u^+(r) = \sqrt{\frac{1}{2\,\pi}\,\frac{1+r^2}{1-r^2}}\,\sqrt{\int_0^{2\,\pi} f^2(\theta)\, d\theta} \tag{4}$$

and

$$u^-(r) = \frac{8}{\pi}\,\frac{1-r}{1+r}\,\mathbf{K}^2\left(\frac{2\sqrt{r}}{1+r}\right)\left[\int_0^{2\,\pi} \frac{d\theta}{f(\theta)}\right]^{-1} \tag{5}$$

($\boldsymbol{K}(k)$ is complete elliptic integral of the first kind).

One can easily prove this theorem applying to the Poisson formula (2) direct and reverse Hölder inequalities [2].

It should be emphasized that integral quantities are essential for the construction of estimations (4) and (5), but not the entire boundary condition $f(\theta)$ as a whole. Also it is necessary to note that in formula (5) special function $\mathbf{K}(k)$ may be replaced by its lower bound, expressed in terms of elementary functions and established in article [3].

In the report the double-sided bound (3) is checked numerically for the next boundary condition:

$$f(\theta) = \frac{1-a^2}{1-2\,a\,\cos\theta + a^2}\,, \qquad 0 < a < 1\,, \tag{6}$$

namely, substituting function (6) into integral (2), one can find exact solution of the problem (1) in the explicit form:

$$u(r,\theta) = \frac{1-a^2\,r^2}{1-2\,a\;r\cos\theta + a^2\,r^2}\,. \tag{7}$$

On the other side, for function (6) upper bound (4) and lower one (5) of the solution (7) are reduced to the following axially symmetric functions:

$$u^+(r) = \sqrt{\frac{1+r^2}{1-r^2}}\,\sqrt{\frac{1+a^2}{(1-a^2)^3}} \tag{8}$$

and

$$u^-(r) = \frac{4}{\pi^2}\,\frac{1-r}{1+r}\,\mathbf{K}^2\left(\frac{2\sqrt{r}}{1+r}\right)\frac{1+a^2}{1-a^2}\,. \tag{9}$$

In the work graphs of relative errors for functions (7), (8) and (9) are presented.

The obtained results are also planned to be applied to the evaluation of static solutions of Dirichlet problems for KPZ-type equations:

$$\frac{\partial^2 h}{\partial r^2} + \frac{1}{r}\frac{\partial h}{\partial r} + \frac{1}{r^2}\frac{\partial^2 h}{\partial \theta^2} + \frac{1}{2}\left[\left(\frac{\partial h}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial h}{\partial \theta}\right)^2\right] = 0\,, \qquad h|_{r=1} = H(\theta)\,,$$

and

$$\frac{\partial^2 h}{\partial r^2} + \frac{1}{r}\frac{\partial h}{\partial r} + \frac{1}{r^2}\frac{\partial^2 h}{\partial \theta^2} + \frac{1}{h}\left[\left(\frac{\partial h}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial h}{\partial \theta}\right)^2\right] = 0\,, \qquad h|_{r=1} = H(\theta)\,,$$

on the unit circle, which are reduced to the problem (1) by nonlinear substitutions of unknown functions $h = 2\,\ln u$ and $h = \sqrt{u}$ respectively.

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Pseudohyperbolic attractors of three-dimensional maps

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According to the work of Turaev-Shilnikov [1] the pseudohyperbolicity of the attractor is preserved for small time-periodic perturbations of the system. Using the example of a periodically perturbed Shimizu-Morioka system of the form

$$\begin{cases} \dot{x} = y \\ \dot{y} = x - \lambda y - xz \\ \dot{z} = -\alpha z + x^2 + \varepsilon z \sin t. \end{cases} \tag{1}$$

it is shown that the pseudohyperbolicity of the Lorentz attractor is preserved even under considerable perturbations. The results of numerical studies show that the pseudohyperbolicity of the attractor breaks down at approximately $\varepsilon \approx 0.045$. It is established that the corresponding discrete attractor is wild, i.e. admits the existence of homoclinic tangents.

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On Embedding of two-dimensional Separatrices of Saddle Equilibria in Four-dimensional Manifolds

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Let M^n be a closed smooth manifold. A smooth flow $f^t : M^n \to M^n$ is called *gradient-like* if its non-wandering set Ω_{f^t} consists of finite number of hyperbolic equilibrium states and invariant manifolds of equilibria intersect each other transversely. Suppose that p is a saddle equilibria, $\dim W^u_p \geq 2$ and W^u_p does not intersect stable manifolds of other saddle points. Then there exist a unique sink equilibrium ω such that $\operatorname{cl} W^u_p = W^u_p \cup \omega$. Hence $\operatorname{cl} W^u_p$ is a sphere smoothly embedded in M^n in all points except, maybe, a point ω. In [Proposition 3.1] [1] it is shown that if $\dim W^u_p = n-1$ then $\operatorname{cl} W^u_p$ is locally flat in ω. If $\dim W^u_p = n-2$, then $\operatorname{cl} W^u_p$ may be wild in ω, as it shown in [2].

In the report, a class $G(M^n)$ of gradient-like flows on M^n is considering, in assumption that $n \geq 4$, all $(n-1)$-dimensional invariant manifolds does not contain orbits joining two saddles, and non-wandering set Ω_{f^t} contains exactly one saddle equilibria p such that $\dim W^u_p \notin \{1, n-1\}$. It follows from [3] that $n \in \{4, 8, 16\}$ and $\dim W^u_p = n/2$.

We prove that $\operatorname{cl} W^u_p, \operatorname{cl} W^s_p$ are locally flat in M^n, that makes possible to classify such flows for the case $n = 4$ in combinatorial terms.

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Influence and suppression of noise in deep and echo state neural networks

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Over the past few years, artificial neural networks (ANNs) have found their application in solving many problems from pattern recognition to climate prediction. Nowadays, neural networks have achieved superhuman performance in computational problems previously considered intractable by computers. Depending on the structure of connections within them, several types of ANNs can be used. This article examines two types based on fundamentally different communication patterns: 1 – deep neural networks (feedforward), 2 – recurrent neural networks.

Despite the existence of high-power computing clusters with the ability of paralleling the calculations, neural network modelling on digital equipment is a bottleneck in network scaling, speed of information acquisition and processing, and energy efficiency. This has already led to an exponential increase in the number of works with hardware realised ANNs, which are based on lasers, memristors, spin-transfer oscillators, etc.

The topic of this study is the influence of noise on ensembles of oscillators is not new. However, establishing the fundamental features of noise propagation in neural networks both during the learning process and during operation is a fundamentally new direction. As a rule, there are works on processing a noisy input signal by a neural network. In the case of implementing hardware models of neural networks in an experiment, noise can be internal, that is, it can be located both inside the neuron and inside the connection.

Here we propose methods for estimating the noise level of the output signal of a neural network and ways to reduce it using special connections between neurons. Several types of noise are considered: additive and multiplicative, depending on how the noise affects one individual neuron, and correlated and uncorrelated noise, depending on how the noise affects a group of neurons.

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Multi-dimensional chaos in the systems with impulse impact

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In the work we consider three-dimensional flow dynamical systems under action of periodic sequence of pulsed force. Stabilization by a periodic pulsed force of trajectories running away to infinity in the Rössler system at a threshold of a saddle-node bifurcation (birth of equilibrium states) is studied. It is shown that the external pulsed action stabilizes dynamical regimes in a fairly wide range of external signal parameters. Stabilized regimes can be periodic, quasi-periodic, or chaotic. Three types of chaotic oscillations are revealed depending on the spectrum of Lyapunov exponents: the simplest (classical) and multi-dimensional (hyperchaos and chaos with additional zero Lyapunov exponent). Scenarios for the development of multi-dimensional chaos have been studied in detail. The universality of the observed behavior when changing the direction of the external force is investigated. It is shown that the effects are universal, when force acts in a plane corresponding to focal behavior of the trajectories, stabilization is not observed if the direction of force is perpendicular to the plane. The universality of the obtained picture is studied when the autonomous dynamics of the model change, it is shown that for small periods of external action, the picture is determined by a transient processes of autonomous model and remains characteristic for stabilization. With an increase in the period of external force, the properties of an autonomous model appear.

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Instability and non-uniqueness for Navier-Stokes-Brinkman-Forchheimer equations

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We consider the following version of the Navier-Stokes-Brinkman-Forchheimer equation in the whole space $x \in \mathbb{R}^3$:

$$\partial_t u + (u, \nabla)u + \nabla p + \varepsilon u|u|^3 = \nu\Delta u + g(t), \quad \operatorname{div} u = 0, \ u\big|_{t=0} = u_0,$$

where $u = (u^1, u^2, u^3)$ and p are unknown velocity and pressure respectively, $\nu > 0$ and $\varepsilon > 0$ are given kinematic viscosity and the Forchheimer parameter and $g(t)$ are given external forces which are singular at $t = 0$, but this singularity is mild enough to guarantee the existence of weak energy solutions and even the validity of the energy equality.

It is shown that, for the case $\varepsilon > \varepsilon_0(\nu)$, the weak energy solution is unique, but this solution may be not unique if $\varepsilon > 0$ is small enough. The non-uniqueness result is obtained similarly to the case of forced Navier-Stokes equation (which corresponds to the case $\varepsilon = 0$) and is based on the instability of the so-called Vishik vortices.

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Reversible homoclinic implies chaos

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We consider reversible vector fields such that the dimension of the set of fixed points of the involutory reversing symmetry is half the dimension of the phase space. Let such system have a smooth one-parameter family of symmetric periodic orbits which is of saddle type in normal directions. We establish that topological entropy is positive when the stable and unstable manifolds of this family of periodic orbits have a strongly transverse intersection. This is a joint work with Jeroen lamb and Ale Jan Homburg.

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